Negative order MKdV hierarchy and a new integrable Neumann-like system

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Abstract

The purpose of this paper is to develop the negative order MKdV hierarchy and to present a new related integrable Neumann-like Hamiltonian flow from the view point of inverse recursion operator and constraint method. The whole MKdV hierarchy both positive and negative is generated by the kernel elements of Lenard's operators pair and recursion operator. Through solving a key operator equation, the whole MKdV hierarchy is shown to have the Lax representation. In particular, some new integrable equation together with the Liouville equations, the sine-Gordon equation, and the sinh-Gordon equation are derived from the negative order MKdV hierarchy. It is very interesting that the restricted flow, corresponding to the negative order MKdV hierarchy, is just a new kind of Neumann-like system. This new Neumann-like system is obtained through restricting the MKdV spectral problem onto a symplectic submanifold and is proven to be completely integrable under the Dirac-Poisson bracket, which we define on the symplectic submanifold. Finally, with the help of the constraint between the Neumann-like system and the negative order MKdV hierarchy, all equations in the hierarchy are proven to have the parametric representations of solutions. In particular, we obtain

the parametric solutions of the sine-Gordon equation and the sinh-Gordon equation.

Keywords Negative order, Positive order, MKdV hierarchy, Lax representation, Neumann-like system, parametric solution.

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1 Introduction

It is well-known that the nonlinear evolution equations (NLEEs) solved by the famous inverse scattering transformation (IST) can be understood as compatible conditions of some linear equations [32, 1], namely, Lax representation. In the past two decades the Lax representation has played a very important role in the discussion of NLEEs. In particular, the Lax representation has been used successively in the bi-Hamiltonian structure of finite-dimensional dynamical systems [8, 5], in the nonlinearization theory of soliton system to produce new completely integrable systems in the Liouville sense [11, 12], in the tri-Hamiltonian formulation of nonlinear equations [16], and in the finite-dimensional restricted flows of the underlying infinite systems [6]. Recently, the finding of peaked solitons produced a breakthrough in the study of nonlinear partial differential equations [9], where Camassa and Holm showed their equation is completely integrable by the IST method. This fact allows one to discuss the nonlinear dynamics of soliton solutions and billiard solution [4] via their linear spectral content (i.e. Lax representation). Thus, for a given hierarchy of NLEEs, to find the Lax representation is of great importance.

The modified Korteweg-de Vries (MKdV) equation together with the MKdV hierarchy has wide applications in physics and other nonlinear sciences. It possesses the Lax pair [1], the periodic soliton solution [30, 31], the bi-Hamiltonian structure [14] and other soliton properties such as Darboux transformation, Bäcklund transformation and the Miura transformation between it and the KdV equation [20]. About the study of the MKdV hierarchy, i.e. usual higher-order MKdV equations, there have been many discussions in the literature. In Ref. [15], the sine-Gordon equation

and the intergrated MKdV equations were shown to conserve the same infinite set of charges which were determined by a recursion relation. Afterward, Verosky [29] introduced the negative powers of Olver recursion operator and presented the relations between the sine-Gordon/sinh-Gordon equation and the potential MKdV equations (non-local flows). In 1993, Andree and Shmakova [2] discussed the supersymmetry structure of the sine-Gordon equation, which can be embraced in the MKdV and KdV hierarchies. All of those interesting facts happened between the sine-Gordon equation and the MKdV equation. The fact that the sine-Gordon equation is a negative flow of MKdV can be seen in Ref. [32].

In recent years, the time-discrete version [13] of integrable systems have already arisen a lot of attractive attentions. This idea was applied to constructing a new Lax pair from the old Lax pair [22]. In Ref. [23], we proposed an approach to generate the positive and the negative order integrable hierarchies from a given spectral problem. In fact, we had this idea in Ref. [25]. But that depends on the existence of inverse recursion operator. This means, for every concrete spectral problem, we need to determine the inverse of recursion operator.

In the present paper, we will give the inverse of recursion operator of the MKdV hierarchy in an explicit form. With the help of the recursion operator and its inverse, we present the positive order and the negative order MKdV hierarchies of NLEEs. The whole MKdV hierarchy is proven to have the Lax pair through employing a key operator equation, and is therefore a completely integrable hierarchy. Particularly, some new integrable equation together with the Liouville equations, the sine-Gordon equation, and the sinh-Gordon equation are derived from the negative order MKdV hierarchy. Furthermore, the constraint between the MKdV spectral problem and the negative order MKdV hierarchy derives a restricted Hamiltonian flow on a symplectic submanifold, which is a new kind of Neumann-like system. Then, we define the Dirac-Poisson bracket on the symplectic submanifold, which is a very useful tool to deal with the finite dimensional integrable system on some submanifols. Under the Dirac-Poisson bracket, the new Neumann-like system has the Hamiltonian canonical form and has furthermore independent and involutive functions, which guarantees its complete integrability. Finally, each equation in the negative order MKdV hierarchy is proven to have the parametric representation of solution, which is given by the involutive solution of Hamiltonian phase flows (i.e. x-flow and t_m-flow, $m < 0, m \in$

$\mathbb{Z}$). Particularly, we obtain the parametric solutions of the sine-Gordon equation and the sinh-Gordon equation.

The whole paper is organized as follows. Next section gives the pair of Lenard's operators, the recursion operator and their inverses, and introduces a key operator equation which is available for both the positive order and the negative order MKdV hierarchies. This operator equation is different from the one usually considered in literatures [10, 26, 27]. In section 3, we will see that the positive order MKdV hierarchy, arising from the kernel of one of Lenard's operators, is nothing but the well-known MKdV hierarchy. In section 4, we present the negative order MKdV hierarchy by using the kernel element of the other one of Lenard's operators. All equations in the hierarchy have the Lax pairs. In sections 5 and 6, we provide a new kind of integrable Neumann-like system on a symplectic submanifold, and give the parametric solutions of the negative order MKdV hierarchy, respectively. In the last section, we give some conclusions.

For convenience, we make the following conventions:

$$f^{(m)} = \begin{cases} \frac{\partial^m}{\partial x^m} f = f_{mx}, & m = 0, 1, 2, ..., \\ \underbrace{\int \ldots \int}_{-m} f dx, & m = -1, -2, ..., \end{cases}$$

$f_t = \frac{\partial f}{\partial t}$, $f_{mxt} = \frac{\partial^{m+1} f}{\partial t \partial x^m}$ $(m = 0, 1, 2, ...)$, $\partial = \frac{\partial}{\partial x}$, ∂^{-1} is the inverse of ∂, i.e. $\partial\partial^{-1} = \partial^{-1}\partial = 1$, $\partial^m f$ means the operator $\partial^m f$ acts on some function g, i.e.

$$\partial^m f \cdot g = \partial^m (fg) = \begin{cases} \frac{\partial^m}{\partial x^m}(fg) = (fg)_{mx}, & m = 0, 1, 2..., \\ \underbrace{\int \ldots \int}_{-m} fg dx, & m = -1, -2, ... \end{cases}$$

In the following the function u stands for potential, the imaginary unit i is satisfying $i^2 = -1$, and λ is assumed to be a spectral parameter, and the domain of the spatial variable x is Ω which becomes equal to $(-\infty,\ +\infty)$ or $(0,\ T)$, while the domain of the time variable t_m is the positive time axis $\mathbb{R}^+ = \{t_m |\ t_m \in \mathbb{R},\ t_m \geq 0,\ m = 0, \pm 1, \pm 2, ...\}$. In the case $\Omega = (-\infty,\ +\infty)$, the decaying condition at infinity and in the case $\Omega = (0,\ T)$, the periodicity condition for the potential function is imposed.

$(\mathbb{R}^{2N}, dp\wedge dq)$ stands for the standard symplectic structure in Euclid space $\mathbb{R}^{2N} = \{(p,q)|\ p=(p_1,\ldots,p_N), q=(q_1,\ldots,q_N)\}$, p_j, q_j $(j=1,\ldots,N)$ are N pairs of canonical coordinates, $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbb{R}^N$; in $(\mathbb{R}^{2N}, dp\wedge dq)$, the Poisson bracket of two Hamiltonian functions F, H is defined by [7]

$$\{F,H\} = \sum_{j=1}^{N}\left(\frac{\partial F}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial H}{\partial q_j}\right) = \left\langle \frac{\partial F}{\partial q}, \frac{\partial H}{\partial p}\right\rangle - \left\langle \frac{\partial F}{\partial p}, \frac{\partial H}{\partial q}\right\rangle. \tag{1.1}$$

$\lambda_1, ..., \lambda_N$ are N distinct spectral parameters, and $\Lambda = diag(\lambda_1, ..., \lambda_N)$. Denote all infinitely times differentiable functions on real field $\mathbb{R}$ and all integers by $C^\infty(\mathbb{R})$ and by $\mathbb{Z}$, respectively.

2 Inverse recursion operator and operator equation

Let us consider the following spectral problem

$$\psi_x = \begin{pmatrix} -i\lambda & u \\ u & i\lambda \end{pmatrix}\psi, \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{2.1}$$

which is quite a special case of the well-known Zakharov-Shabat-AKNS spectral problem [33]

$$\psi_x = \begin{pmatrix} -i\lambda & u \\ v & i\lambda \end{pmatrix}\psi \tag{2.2}$$

with $v = u$.

Eq. (2.1) is equivalent to

$$L\cdot\psi \equiv \begin{pmatrix} i\partial & -iu \\ iu & -i\partial \end{pmatrix}\cdot\psi = \lambda\psi \tag{2.3}$$

and its spectral gradients $\nabla\lambda \equiv \frac{\delta\lambda}{\delta u} = \psi_2^2 - \psi_1^2$ satisfies the Lenard eigenvalue problem

$$K\cdot\nabla\lambda = \lambda^2 J\cdot\nabla\lambda \tag{2.4}$$

with the Lenard's operators pair

$$K=-\frac{1}{4}\partial^3+\partial u\partial^{-1}u\partial,\quad J=\partial, \tag{2.5}$$

which yields the recursion operator

$$\mathcal{L}=J^{-1}K=-\frac{1}{4}\partial^2+u\partial^{-1}u\partial. \tag{2.6}$$

Apparently, the Gateaux derivative operator $L_*(\xi)$ of the spectral operator L given by Eq. (2.3) in the direction $\xi\in C^\infty(\mathbb{R})$ is

$$L_*(\xi)\triangleq\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}L(u+\epsilon\xi)=\begin{pmatrix}0 & -i\xi\\ i\xi & 0\end{pmatrix} \tag{2.7}$$

which is obviously an injective homomorphism.

Through guesswork and calculations, we can obtain the inverse operators of $L,\ J,\ K$ and $\mathcal{L}$:

$$\begin{aligned}
L^{-1} &= \begin{pmatrix}A & -\partial^{-1}uA\\ \partial^{-1}uA & -A\end{pmatrix}, && (2.8)\\
& \quad A=-ie^{u^{(-1)}}\partial^{-1}e^{-2u^{(-1)}}u\partial^{-1}e^{u^{(-1)}}\partial u^{-1}, &&\\
J^{-1} &= \partial^{-1}, && (2.9)\\
K^{-1} &= -4\partial^{-1}e^{-2u^{(-1)}}\partial^{-1}e^{4u^{(-1)}}u\partial^{-1}e^{-2u^{(-1)}}\partial u^{-1}\partial^{-1}, && (2.10)\\
\mathcal{L}^{-1} &= -4\partial^{-1}e^{-2u^{(-1)}}\partial^{-1}e^{4u^{(-1)}}u\partial^{-1}e^{-2u^{(-1)}}\partial u^{-1}. && (2.11)
\end{aligned}$$

For any given C^∞-function G, we construct the following operator equation with respect to $V=V(G)$

$$[V,L]\ =\ L_*(K\cdot G)-L_*(J\cdot G)L^2. \tag{2.12}$$

Remark 1 *This equation contains a special term L^2 instead of the term L usually considered in literatures [10, 26, 27].*

Theorem 1 *For the MKdV spectral operator (2.3) and an arbitrarily given C^∞-function G, the operator equation (2.12) has the following solution*

$$V = V(G) = \begin{pmatrix} (uG_x)^{(-1)}\partial - \frac{1}{2}uG_x & \frac{1}{2}G_x\partial - \frac{1}{4}G_{xx} \\ \frac{1}{2}G_x\partial - \frac{1}{4}G_{xx} & (uG_x)^{(-1)}\partial - \frac{1}{2}uG_x \end{pmatrix}. \tag{2.13}$$

Proof: Directly substituting Eqs. (2.13), (2.3), (2.5) and (2.7) into Eq. (2.12), we can complete the proof of this theorem.

3 Positive order hierarchy of Eq. (2.1), i.e. usual MKdV hierarchy

Let us now give the positive order MKdV hierarchy through considering the kernel element of the Lenard's operator J.

$G_0 = a \in KerJ$ and the recursion operator (2.6) yield the positive order hierarchy of Eq. (2.3)

$$u_{t_m} = J\mathcal{L}^m \cdot a, \quad m = 0, 1, 2, \ldots, \tag{3.1}$$

which has the following representative equations

$$u_{t_1} = au_x, \quad \text{trivial case,} \tag{3.2}$$

$$u_{t_2} = -\frac{1}{4}au_{xxx} + \frac{3}{2}au^2u_x. \tag{3.3}$$

Here, $a = a(t_n) \in C^\infty(\mathbb{R})$ is an arbitrarily given fuction with respect to variables t_n $(n \geq 0, n \in \mathbb{Z})$, but independent of x. Apparently, with $a = 4$ Eq. (3.3) becomes the well-known MKdV equation

$$u_{t_2} - 6u^2u_x + u_{xxx} = 0. \tag{3.4}$$

Therefore, Eq. (3.1) coincides with the well-known MKdV hierarchy, which corresponds to the isospectral case: $\lambda_{t_m} = 0$.

By Eq. (2.13), the whole hierarchy (3.1) has the standard Lax representation

$$L_{t_m} = [W_m, L], \tag{3.5}$$

$$W_m = \sum_{j=0}^{m-1} V(G_j) L^{2(m-j-1)}, \tag{3.6}$$

where $V(G_j)$ is given by Eq. (2.13) with $G = G_j = \mathcal{L}^j \cdot a,\ j \geq 0,\ j \in \mathbb{Z}$. Therefore we obtain the following theorem.

Theorem 2 *The positive order hierarchy (3.1) (i.e. the MKdV hierarchy) of the spectral problem (2.3) possesses the Lax pair*

$$\begin{cases} \psi_x = \begin{pmatrix} -i\lambda & u \\ u & i\lambda \end{pmatrix} \psi, \\ \psi_{t_m} = a \begin{pmatrix} -i\lambda & u \\ u & i\lambda \end{pmatrix} \lambda^{2(m-1)}\psi + \sum_{j=1}^{m-1} V_j \lambda^{2(m-j-1)}\psi, \quad m = 0, 1, 2, ..., \end{cases} \tag{3.7}$$

where

$$V_j = \begin{pmatrix} -i\lambda(uG_{j,x})^{(-1)} & \frac{1}{2}i\lambda G_{j,x} - \frac{1}{4}G_{j,xx} + u(uG_{j,x})^{(-1)} \\ -\frac{1}{2}i\lambda G_{j,x} - \frac{1}{4}G_{j,xx} + u(uG_{j,x})^{(-1)} & i\lambda(uG_{j,x})^{(-1)} \end{pmatrix}, \tag{3.8}$$

$$G_j = \mathcal{L}^j \cdot a,\ j \geq 0,\ j \in \mathbb{Z}. \tag{3.9}$$

In particular, the MKdV equation (3.3) has the Lax pair

$$\begin{cases} \psi_x = \begin{pmatrix} -i\lambda & u \\ u & i\lambda \end{pmatrix} \psi, \\ \psi_{t_2} = a \begin{pmatrix} -i(\lambda^3 + \frac{1}{2}\lambda u^2) & \lambda^2 u + \frac{1}{2}i\lambda u_x - \frac{1}{4}u_{xx} + \frac{1}{2}u^3 \\ \lambda^2 u - \frac{1}{2}i\lambda u_x - \frac{1}{4}u_{xx} + \frac{1}{2}u^3 & i(\lambda^3 + \frac{1}{2}\lambda u^2) \end{pmatrix} \psi. \end{cases} \tag{3.10}$$

Remark 2 *For the spectral problem (2.1), if we use the usual method [33, 28], i.e. the method of finite power expansion with respect to spectral parameter λ, then no isospectral evolution equations of Eq. (2.1) can be obtained.*

However, we here present the MKdV hierarchy (3.1) purely by the Lenard's operators pair satisfying Eq. (2.4). Due to containing the spectral gradient $\nabla\lambda$ in Eq. (2.4), this procedure of generating evolution equations from a given spectral problem is called the spectral gradient method.

In this method, how to determine a pair of Lenard's operators associated with the given spectral problem mainly depends on the concrete forms of spectral problems and spectral gradients, and some computational techniques. From this method, we can furthermore derive the negative order MKdV hierarchy, which is displayed below.

4 The negative order MKdV hierarchy and Lax representation

Let us now give the negative order MKdV hierarchy through considering the kernel element of Lenard's operator K. The kernel of operator K has the following three seed functions:

$$\bar{G}^1_{-1} = \left(e^{-2u^{(-1)}}\right)^{(-1)}, \tag{4.1}$$

$$\bar{G}^2_{-1} = \left(e^{2u^{(-1)}}\right)^{(-1)}, \tag{4.2}$$

$$\bar{G}^3_{-1} = \left(e^{-2u^{(-1)}}\right)^{(-1)}\left(e^{2u^{(-1)}}\right)^{(-1)}, \tag{4.3}$$

whose all possible linear combinations form the whole kernel of K. Let $\bar{G}_{-1} \in Ker\ K$, then

$$\bar{G}_{-1} = \sum_{k=1}^{3} a_k \bar{G}^k_{-1} \tag{4.4}$$

where $a_k = a_k(t_n),\ k = 1, 2, 3$, are three arbitrarily given C^∞-functions with respect to variables t_n ($n < 0, n \in \mathbb{Z}$), but independent of x. Therefore, $\bar{G}_{-1}$ directly generates the isospectral ($\lambda_{t_m} = 0$) negative order hierarchy of nonlinear evolution equations for the spectral problem (2.3)

$$u_{t_m} = J\mathcal{L}^{m+1} \cdot \bar{G}_{-1}, \quad m < 0,\ m \in \mathbb{Z}, \tag{4.5}$$

which is called the negative order MKdV hierarchy of Eq. (2.3). By Theorem 1, the hierarchy (4.5) has the standard Lax representation

$$L_{t_m} = \left[\bar{W}_m, L\right] \tag{4.6}$$

$$\bar{W}_m = -\sum_{j=m}^{-1} V\left(\bar{G}_j\right) L^{2(m-j-1)}, \quad m=-1,-2,..., \tag{4.7}$$

i.e.

$$\begin{cases} \psi_x = \begin{pmatrix} -i\lambda & u \\ u & i\lambda \end{pmatrix} \psi, \\ \psi_{t_m} = \sum_{j=m}^{-1} \bar{V}_j \lambda^{2(m-j-1)} \psi, \quad m=-1,-2,..., \end{cases} \tag{4.8}$$

with

$$\bar{V}_j = \begin{pmatrix} i\lambda\left(u\bar{G}_{j,x}\right)^{(-1)} & -\frac{1}{2}i\lambda\bar{G}_{j,x} - \mathcal{L}\cdot\bar{G}_j \\ \frac{1}{2}i\lambda\bar{G}_{j,x} - \mathcal{L}\cdot\bar{G}_j & -i\lambda\left(u\bar{G}_{j,x}\right)^{(-1)} \end{pmatrix}, \tag{4.9}$$

where

$$\mathcal{L}\cdot\bar{G}_j = -\frac{1}{4}\bar{G}_{j,xx} + u\left(u\bar{G}_{j,x}\right)^{(-1)}.$$

In Eq. (4.7), $V\left(\bar{G}_j\right)$ and L^{-1} are given by Eq. (2.13) with $G = \bar{G}_j = \mathcal{L}^{j+1}\cdot\bar{G}_{-1}$ and by Eq. (2.8), respectively. Thus, all nonlinear equations in the hierarchy (4.5) are integrable.

Let us now give some special reductions of Eq. (4.5).

- In the cases of $a_2 = a_3 = 0$; $a_1 = a_3 = 0$; $a_1 = a_2 = 0$, Eq. (4.5) separately has the following representative equations

$$u_{t_{-1}} = a_1 e^{-2u^{(-1)}}, \tag{4.10}$$

$$u_{t_{-1}} = a_2 e^{2u^{(-1)}}, \tag{4.11}$$

$$u_{t_{-1}} = a_3\left(e^{-2u^{(-1)}}\left(e^{2u^{(-1)}}\right)^{(-1)} + e^{2u^{(-1)}}\left(e^{-2u^{(-1)}}\right)^{(-1)}\right), \tag{4.12}$$

which can be via the transformation $u = v_x$ respectively changed to

$$v_{x,t_{-1}} = a_1 e^{-2v}, \quad \text{Liouville equation}, \tag{4.13}$$

$$v_{x,t_{-1}} = a_2 e^{2v}, \quad \text{Liouville equation}, \tag{4.14}$$

$$v_{t_{-1}} = a_3\left(e^{2v}\right)^{(-1)}\left(e^{-2v}\right)^{(-1)}, \quad \text{a new integrable equation}. \tag{4.15}$$

They are also equivalent to the following differential equations:

$$u_{x,t_{-1}} + 2uu_{t_{-1}} = 0, \tag{4.16}$$
$$u_{x,t_{-1}} - 2uu_{t_{-1}} = 0, \tag{4.17}$$
$$u_{xx,t_{-1}} - u^{-1}u_x u_{x,t_{-1}} + u^{-1}u_x - 4u^2 u_{t_{-1}} = 0. \tag{4.18}$$

Apparently, Eqs. (4.10), (4.11), and (4.12) possess the following standard Lax pairs

$$\begin{cases} \psi_x = \begin{pmatrix} -i\lambda & u \\ u & i\lambda \end{pmatrix} \psi, \\ \psi_{t_{-1}} = \bar{W}_{-1}^1 \cdot \psi = \frac{1}{2} i a_1 e^{-2u^{(-1)}} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \lambda^{-1}\psi, \end{cases} \tag{4.19}$$

$$\begin{cases} \psi_x = \begin{pmatrix} -i\lambda & u \\ u & i\lambda \end{pmatrix} \psi, \\ \psi_{t_{-1}} = \bar{W}_{-1}^2 \cdot \psi = \frac{1}{2} i a_2 e^{2u^{(-1)}} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \lambda^{-1}\psi, \end{cases} \tag{4.20}$$

$$\begin{cases} \psi_x = \begin{pmatrix} -i\lambda & u \\ u & i\lambda \end{pmatrix} \psi, \\ \psi_{t_{-1}} = \bar{W}_{-1}^3 \cdot \psi = 2a_3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \lambda^{-2}\psi + U_{-1}\lambda^{-1}\psi, \end{cases} \tag{4.21}$$

respectively, where

$$\begin{aligned} U_{-1} &= 2ia_3 e^{2u^{(-1)}} \left(e^{-2u^{(-1)}}\right)^{(-1)} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \\ &\quad + 2ia_3 e^{-2u^{(-1)}} \left(e^{2u^{(-1)}}\right)^{(-1)} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}. \end{aligned} \tag{4.22}$$

Eqs. (4.16) and (4.17) are two Liouville equations, and easy to see that they can be directly integrated as $u^2 \pm u_x = f(x),\ f(x) \in C^\infty(\mathbb{R})$, which are two typical Riccati equations. They can be solved by some methods in the theory of ordinary differential equations. But, Eq. (4.15) or (4.18) is a new integrable equation.

- In the case of $a_1 = -\frac{1}{4}$, $a_2 = \frac{1}{4}$, and $a_3 = 0$, the first equation of Eq. (4.5) reads

$$u_{t_{-1}} = \frac{e^{2u^{(-1)}} - e^{-2u^{(-1)}}}{4}. \tag{4.23}$$

We make a simple transformation

$$u = \frac{1}{2} i v_x \tag{4.24}$$

then Eq. (4.23) is exactly changed to **the well-known sine-Gordon equation**

$$v_{x,t_{-1}} = \sin v. \tag{4.25}$$

According to Eq. (4.8), the sine-Gordon equation (4.25) possesses the following Lax pair:

$$\begin{cases} \psi_x = \begin{pmatrix} -i\lambda & \frac{1}{2} i v_x \\ \frac{1}{2} i v_x & i\lambda \end{pmatrix} \psi, \\ \psi_{t_{-1}} = \frac{1}{4\lambda} \begin{pmatrix} i\cos v & \sin v \\ -\sin v & -i\cos v \end{pmatrix} \psi, \end{cases} \tag{4.26}$$

which has a slight difference from the usual one given in Ref. [32, 17].

For Eq. (4.23), if we make the transformation $u = \frac{1}{2} v_x$, then it becomes

$$v_{x,t_{-1}} = \sinh v \tag{4.27}$$

which is nothing but **the well-known sinh-Gordon equation**. By Eqs. (4.8) and (4.9), the sinh-Gordon equation (4.27) has the following Lax pair

$$\begin{cases} \psi_x = \begin{pmatrix} -i\lambda & \frac{1}{2} v_x \\ \frac{1}{2} v_x & i\lambda \end{pmatrix} \psi, \\ \psi_{t_{-1}} = \frac{1}{4\lambda} i \begin{pmatrix} \cosh v & -\sinh v \\ \sinh v & -\cosh v \end{pmatrix} \psi, \end{cases} \tag{4.28}$$

which is also slightly different from the usual one [32, 17].

Remark 3 *In Ref. [3], Alber, Camassa, Holm and Marsden took evolution equations of auxiliary linear system polynomial in λ^{-1} and gave the zero-curvature representation for the Dym type hierarchy and the Camassa-Holm equation. Here, our starting point is the spectral problem and the spectral gradient instead of the auxiliary linear problem, then via a key operator equation we obtain the Lax pairs.*

5 A new integrable Neumann-like system

In Ref. [24], we presented an integrable Neumann-like system closely associated with the positive order MKdV hierarchy (3.1), showed that the Neumann-like system was sent by a gauge transformation to an integrable Bargmann system, and obtained the parametric solutions for the positive order MKdV hierarchy. Now, we study the negative case.

Let $\lambda_1, ..., \lambda_N$ be N different spectral parameters of Eq. (2.1), $(q_j, p_j)^T$ the spectral function corresponding to λ_j, and $i^2 = -1$. Then Eq. (2.1) becomes

$$\begin{cases} q_{j,x} = -i\lambda_j q_j + u p_j, \\ p_{j,x} = u q_j + i\lambda_j p_j, \end{cases}$$

i.e. Eq. (2.1) yields the following form:

$$\begin{cases} q_x = -i\Lambda q + up, \\ p_x = uq + i\Lambda p, \end{cases} \tag{5.1}$$

where $p = (p_1, ..., p_N)^T,\ q = (q_1, ..., q_N)^T$, and $\Lambda = diag(\lambda_1, ..., \lambda_N)$.

Assume $\langle \Lambda q, p\rangle \neq 0$, then let us restrict Eq. (5.1) onto the following symplectic submanifold $\mathbb{M}$ in $\mathbb{R}^{2N}$

$$\begin{aligned} \mathbb{M} = \Bigg\{ (q,p) \in \mathbb{R}^{2N} \,\Bigg|\, & F \equiv \langle q, q\rangle - \langle p, p\rangle - \frac{1}{4} = 0, \\ & G \equiv \frac{1}{2}\left(\langle \Lambda q, q\rangle + \langle \Lambda p, p\rangle\right) = 0 \Bigg\}. \end{aligned} \tag{5.2}$$

Then, on $\mathbb{M}$ we obtain the following constraint

$$u = i\frac{\langle \Lambda^2 q, q\rangle - \langle \Lambda^2 p, p\rangle}{2\langle \Lambda q, p\rangle}. \tag{5.3}$$

Thus, under Eq. (5.3) the MKdV spectral problem (2.1) is nonlinearized as the following nonlinear system

$$\begin{cases} q_x = -i\Lambda q + i\frac{\langle \Lambda^2 q, q\rangle - \langle \Lambda^2 p, p\rangle}{2\langle \Lambda q, p\rangle} p, \\ p_x = i\frac{\langle \Lambda^2 q, q\rangle - \langle \Lambda^2 p, p\rangle}{2\langle \Lambda q, p\rangle} q + i\Lambda p, \end{cases} \tag{5.4}$$

which is called **a restricted MKdV flow of the spectral problem (2.1) on $\mathbb{M}$.**

Remark 4 *Because dF, dG are independent everywhere on $\mathbb{M}$ and their determinant* $\det(\{F,G\}) = \langle \Lambda q, p\rangle \neq 0$, *$\mathbb{M}$ is therefore a symplectic submanifold in $\mathbb{R}^{2N}$ [17]. Apparently, $\mathbb{M}$ is not the usual tangent bundle, i.e. $\mathbb{M} \neq TS^{N-1} = \Big\{ (q,p) \in \mathbb{R}^{2N} \Big|\ \tilde{F} \equiv \langle q,q\rangle - 1 = 0, \tilde{G} \equiv \langle q,p\rangle = 0 \Big\}$, thus Eq. (5.3) does not coincides with the usual Neumann constraint [19, 18] on TS^{N-1}. If we strongly impose Eq. (5.1) on the usual tangent bundle $TS^{N-1} = \Big\{ (q,p) \in \mathbb{R}^{2N} \Big|\ \tilde{F} \equiv \langle q,q\rangle - 1 = 0,$ $\tilde{G} \equiv \langle q,p\rangle = 0\}$, then we have no constraints except for $u = 0$ which is of course meaningless. So, Eq. (5.4) has no link to the standard Neumann system and is therefore a new kind of Neumann-like system.*

In order to prove the integrability of the restricted flow (5.4) on $\mathbb{M}$, we introduce the Dirac bracket

$$\{f,g\}_D = \{f,g\} + \frac{1}{2\langle \Lambda q, p\rangle}\left(\{f,F\}\{G,g\} - \{f,G\}\{F,g\}\right) \tag{5.5}$$

which is easily proven to be bilinear, skew-symmetric and satisfy the Jacobi identity.

Let us now consider a very simple Hamiltonian function

$$H = -i\langle \Lambda q, p\rangle \tag{5.6}$$

together with independent functions

$$\begin{aligned} F_m \;=\; & \frac{1}{8}\left(\left\langle \Lambda^{2m} p, p\right\rangle - \left\langle \Lambda^{2m} q, q\right\rangle\right) \\ & + \frac{1}{4}\sum_{j=m}^{-2}\left(\left\langle \Lambda^{2(j+1)} q, q\right\rangle - \left\langle \Lambda^{2(j+1)} p, p\right\rangle\right)\left(\left\langle \Lambda^{2(m-j-1)} p, p\right\rangle - \left\langle \Lambda^{2(m-j-1)} q, q\right\rangle\right) \\ & + \frac{1}{4}\sum_{j=m}^{-1}\begin{vmatrix} \langle \Lambda^{2j+1} q, q\rangle + \langle \Lambda^{2j+1} p, p\rangle & 2\left\langle \Lambda^{2(m-j)-1} p, q\right\rangle \\ 2\langle \Lambda^{2j+1} q, p\rangle & \left\langle \Lambda^{2(m-j)-1} q, q\right\rangle + \left\langle \Lambda^{2(m-j)-1} p, p\right\rangle \end{vmatrix}, \\ & m = -1, -2, ... \end{aligned} \tag{5.7}$$

Lemma 1 *The inner product $\left\langle \frac{\partial F_m}{\partial q}, \frac{\partial F_n}{\partial p}\right\rangle$ is symmetric with respect to $m,\ n$ ($m, n < 0, m,\ n \in \mathbb{Z}$).*

Proof: Making the derivatives of F_m with respect to $q,\ p$ and directly substituting them into $\left\langle \frac{\partial F_m}{\partial q}, \frac{\partial F_n}{\partial p} \right\rangle$, we have a lengthy calculation and then know that this inner product is sum of some symmetric terms with respect to $m,\ n$ $(m, n = -1, -2, ...)$.

Proposition 1

$$\{F_m, F_n\} = \{H, F_m\} = 0, \quad m,\ n = -1, -2, ... \tag{5.8}$$

Proof: Lemma 1 directly yields $\{F_m, F_n\} = 0$. As for the second equality, a straightforward computation completes its proof.

Through some guesswork, we find that the restricted flow (5.4) can be expressed as the canonical Hamiltonian form in the following theorem.

Theorem 3 *Under the Dirac-Poisson bracket (5.5), the restricted MKdV flow (5.4) coincides with:*

$$(H)_D: \quad \begin{cases} q_x = \{q, H\}_D, \\ p_x = \{p, H\}_D, \end{cases} \tag{5.9}$$

where H is defined by Eq. (5.6).

A furthermore direct calculation leads to the following lemma.

Lemma 2

$$\{F_m, F_n\}_D = \{H, F_m\}_D = 0, \quad m,\ n = -1, -2, ... \tag{5.10}$$

Because F_m are independent, we obtain the following theorem.

Theorem 4 *The restricted MKdV flow (5.4) on the symplectic submanifold $\mathbb{M}$ is completely integrable. Moreover, all restricted flows $(F_m)_D$ on $\mathbb{M}$*

$$(F_m)_D: \quad \begin{cases} q_{t_m} = \{q, F_m\}_D, \\ p_{t_m} = \{p, F_m\}_D, \end{cases} \tag{5.11}$$

are integrable.

6 Parametric solution of the negative order MKdV hierarchy

In the following, we will consider the relation between constraint and nonlinear equations in the negative order MKdV hierarchy (4.5). Let us start from the following setting

$$\bar{G}_{-1} = \sum_{j=1}^{N} \lambda_j^{-2} \nabla \lambda_j, \tag{6.1}$$

where $\bar{G}_{-1}$ are defined by Eq. (4.4), and $\nabla \lambda_j = p_j^2 - q_j^2$ is the functional gradient of the spectral problem (2.1) corresponding to λ_j.

Let the recursion operator $\mathcal{L}$ act on both sides of Eq. (6.1). Then, through making a choice of $J^{-1} \cdot 0 = \partial^{-1} \cdot 0 = \frac{1}{4}$, we obtain

$$\langle p, p \rangle - \langle q, q \rangle = \frac{1}{4}. \tag{6.2}$$

Doing the derivative on both sides of Eq. (6.2) with respect to x and substituting Eq. (5.1) yield

$$\langle \Lambda q, q \rangle + \langle \Lambda p, p \rangle = 0, \tag{6.3}$$

which together with Eq. (6.2) forms the symplectic submanifold $\mathbb{M}$ we need. Apparently, derivative for Eq. (6.3) with respect to x leads to the constraint relation (5.3).

Since the restricted Hamiltonian flows $(H)_D$ and $(F_m)_D$ are completely integrable and their Poisson brackets $\{H, F_m\}_D = 0$ $(m = -1, -2, ...)$, their phase flows $g_H^x,\ g_{F_m}^{t_m}$ commute [7]. Thus, we can define their compatible solution as follows:

$$\begin{pmatrix} q(x, t_m) \\ p(x, t_m) \end{pmatrix} = g_H^x g_{F_m}^{t_m} \begin{pmatrix} q(x^0, t_m^0) \\ p(x^0, t_m^0) \end{pmatrix}, \quad m = -1, -2, ..., \tag{6.4}$$

where $x^0,\ t_m^0$ are the initial values of phase flows $g_H^x,\ g_{F_m}^{t_m}$.

Theorem 5 *Let $q(x, t_m),\ p(x, t_m)$ be a solution of the compatible Hamiltonian systems $(H)_D$ and $(F_m)_D$ on $\mathbb{M}$. Then*

$$u = i \frac{\langle \Lambda^2 q(x, t_m), q(x, t_m) \rangle - \langle \Lambda^2 p(x, t_m), p(x, t_m) \rangle}{2 \langle \Lambda q(x, t_m), p(x, t_m) \rangle} \tag{6.5}$$

satisfies the negative order MKdV hierarchy

$$u_{t_m} = J\mathcal{L}^{m+1} \cdot \bar{G}_{-1}, \quad m = -1, -2, ... \tag{6.6}$$

Proof: On one hand, the recursion operator $\mathcal{L}$ acts on Eq. (6.1) and results in the following

$$\begin{aligned} J\mathcal{L}^{m+1} \cdot \bar{G}_{-1} &= J \cdot \left(\left\langle \Lambda^{2m} p, p \right\rangle - \left\langle \Lambda^{2m} q, q \right\rangle \right) \\ &= 2 \left(\left\langle \Lambda^{2m} p, p_x \right\rangle - \left\langle \Lambda^{2m} q, q_x \right\rangle \right) \\ &= 2i \left(\left\langle \Lambda^{2m+1} p, p \right\rangle + \left\langle \Lambda^{2m+1} q, q \right\rangle \right). \end{aligned} \tag{6.7}$$

In this procedure, Eqs. (2.4) and (5.4) are used.

On the other hand, we directly make the derivative of Eq. (6.5) with respect to t_m. Then we obtain

$$\begin{aligned} u_{t_m} = \frac{i}{2 \langle \Lambda q, p \rangle^2} & \left(2 \left(\left\langle \Lambda^2 q, q_{t_m} \right\rangle - \left\langle \Lambda^2 p, p_{t_m} \right\rangle \right) \langle \Lambda q, p \rangle \right. \\ & \left. - \left(\left\langle \Lambda^2 q, q \right\rangle - \left\langle \Lambda^2 p, p \right\rangle \right) \left(\langle \Lambda q, p_{t_m} \rangle + \langle \Lambda p, q_{t_m} \rangle \right) \right) \end{aligned} \tag{6.8}$$

where $q = q(x, t_m), \ p = p(x, t_m)$. But,

$$q_{t_m} = \frac{\partial F_m}{\partial p}, \quad p_{t_m} = -\frac{\partial F_m}{\partial q}, \tag{6.9}$$

therefore after substituting them into Eq. (6.8) and calculating it, we have

$$u_{t_m} = 2i \left(\left\langle \Lambda^{2m+1} p, p \right\rangle + \left\langle \Lambda^{2m+1} q, q \right\rangle \right) \tag{6.10}$$

which completes the proof.

In the special case of $m = -1$, we have the following corollary.

Corollary 1 *Let $q(x, t_{-1}), \ p(x, t_{-1})$ be a solution of the compatible Hamiltonian systems $(H)_D$ and $(F_{-1})_D$ on $\mathbb{M}$. Then*

$$u = i \frac{\langle \Lambda^2 q(x, t_{-1}), q(x, t_{-1}) \rangle - \langle \Lambda^2 p(x, t_{-1}), p(x, t_{-1}) \rangle}{2 \langle \Lambda q(x, t_{-1}), p(x, t_{-1}) \rangle} \tag{6.11}$$

is a solution of the nonlinear evolution equation $u_{t_{-1}} = \sum_{k=1}^{3} a_k(t_{-1})\bar{G}^k_{-1,x}$*, i.e.*

$$\begin{aligned} u_{t_{-1}} &= a_1 e^{-2u^{(-1)}} + a_2 e^{2u^{(-1)}} \\ &\quad + a_3 \left(e^{-2u^{(-1)}} \left(e^{2u^{(-1)}} \right)^{(-1)} + e^{2u^{(-1)}} \left(e^{-2u^{(-1)}} \right)^{(-1)} \right), \end{aligned} \tag{6.12}$$

Here H *is defined by Eq. (5.6) and* F_{-1} *is given by*

$$\begin{aligned} F_{-1} &= \frac{1}{8}\left(\left\langle \Lambda^{-2}p, p\right\rangle - \left\langle \Lambda^{-2}q, q\right\rangle\right) \\ &\quad -\frac{1}{4}\left\langle \Lambda^{-1}(q+p), q+p\right\rangle \left\langle \Lambda^{-1}(q-p), q-p\right\rangle . \end{aligned} \tag{6.13}$$

In particular, the Liouville euqations $v_{x,t_{-1}} = a_1 e^{-2v}$, $v_{x,t_{-1}} = a_2 e^{2v}$*; the sine-Gordon equation* $v_{x,t_{-1}} = \sin v$*, and the sinh-Gordon equation* $v_{x,t_{-1}} = \sinh v$ *have the following parametric solution*

$$v = u^{(-1)} = i \int \frac{\langle \Lambda^2 q(x,t_{-1}), q(x,t_{-1})\rangle - \langle \Lambda^2 p(x,t_{-1}), p(x,t_{-1})\rangle}{2\,\langle \Lambda q(x,t_{-1}), p(x,t_{-1})\rangle} dx, \tag{6.14}$$

$$v = -2iu^{(-1)} = \int \frac{\langle \Lambda^2 q(x,t_{-1}), q(x,t_{-1})\rangle - \langle \Lambda^2 p(x,t_{-1}), p(x,t_{-1})\rangle}{\langle \Lambda q(x,t_{-1}), p(x,t_{-1})\rangle} dx, \tag{6.15}$$

and

$$v = 2u^{(-1)} = i \int \frac{\langle \Lambda^2 q(x,t_{-1}), q(x,t_{-1})\rangle - \langle \Lambda^2 p(x,t_{-1}), p(x,t_{-1})\rangle}{\langle \Lambda q(x,t_{-1}), p(x,t_{-1})\rangle} dx, \tag{6.16}$$

respectively.

By Theorem 5, the constraint (5.3) is a solution of the negative order MKdV hierarchy (6.6). Thus, we also **call the system $(H)_D$ (i.e. Eq. (5.4)) a negative order restricted MKdV flow of the spectral problem (2.1) on the symplectic submanifold $\mathbb{M}$. All Hamiltonian systems $(F_m)_D$ (i.e. Eq. (5.11) derived from $(H)_D$) are therefore called the negative order restricted flows on** $\mathbb{M}$.

7 Conclusion and Comparison

It is well-known that some traveling wave solutions or soliton solutions for the integrable equations are available by the Inverse Scattering Transformation. Thus, a

natural question is: what is the relationship between the traveling wave solutions and the parametric solutions (6.5) for the MKdV case? Because we need either the potential function u decaying at $\pm\infty$ or satisfying the periodic condition in the period T with respect to the variable x (see the part of Introduction) when we do the spectral gradient calculations, we believe that both the traveling wave solutions and the periodic or quasi-periodic solutions should be in the formula (6.5), namely, they share a common expression (6.5). In a further procedure, we will consider giving an explicit expression of Eq. (6.5) for the cases of the potential u decaying at $\pm\infty$ or satisfying the periodic condition in the period T.

For the positive order constrained MKdV system by the constraint $u = \langle p, p\rangle - \langle q, q\rangle$, we have dealt with it in Ref. [24] where we knew this constraint is closely connected to the positive order (i.e. usual) MKdV hierarchy (3.1) and in detail discussed the integrability of the constrained flow for the spectral problem (2.1).

A systematic approach to generate new integrable negative order hierarchies of NLEEs can be seen in Ref. [21].

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